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Spin-down of a Boussinesq fluid of small Prandtl number in a circular cylinder

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The impulsive linear spin-down of a stably stratified Boussinesq fluid in a circular cylinder is analyzed under the assumption that the Prandtl number $P = \nu/\kappa$ is small (ν is the kinematic viscosity, κ the thermal diffusivity). The nature of the spin-down process depends on the ordering of P with respect to the Ekman number E. For $P < E^{\frac{1}{2}}$, the spin-down is similar to that of an unstratified fluid. For $P > E^{\frac{1}{2}}$, the process is similar to that for a stratified fluid with $P = O(1)$. The distinctive case $P = O(E^{\frac{1}{2}})$ is analyzed in detail. For that case it is shown that for $N \gg \Omega$, the asymptotic state of rigid rotation is reached in a time of the order of $(N/\Omega)^2\tau$, where N is the Brunt–Väisälä frequency, Ω the angular velocity and τ the thermal diffusion time for the cylinder. We calculate spin-down times for parameter values corresponding to the solar interior. For an angular velocity as large as that suggested by Dicke ($5{\cdot}74 \times 10^{-5}\,\text{sec}^{-1}$) the spin-down time is less than the age of the sun. For an angular velocity comparable to the surface value ($2{\cdot}87 \times 10^{-6}\,\text{sec}^{-1}$), the spin-down time is greater than the age of the sun. These results suggest that a uniformly and rapidly rotating solar interior is not possible, but we cannot rule out a state of non-uniform rotation producing an oblateness as large as that measured by Dicke & Goldenberg (1967).

1. Introduction

An important problem in the theory of rotating fluids is the linear spin-down problem, in which one analyzes the motions induced in an enclosed, uniformly rotating fluid by a change in angular velocity of the container. For an unstratified fluid, the basic parameter is the Ekman number (assumed small),

$$E = \nu/(L^2\Omega), \tag{1}$$

where ν is the kinematic viscosity, L a characteristic container dimension and Ω

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the basic angular velocity. As Greenspan & Howard (1963) showed, the time scale for the interior to reach a state of uniform rotation is

$$t_0 = \Omega^{-1}E^{-\frac{1}{2}}, \tag{2}$$

which is much smaller than the viscous diffusion time $L^2/\nu = t_0 E^{-\frac{1}{2}}$. For a stratified fluid, the Ekman number is still important, and in addition, there is the stratification number

$$S = (N/\Omega)^2, \tag{3}$$

where N is the Brunt–Väisälä frequency. The spin-down of a stratified fluid has been analyzed for a laboratory flow in a circular cylinder by Holton (1965), Walin (1969) and Sakurai (1969*a, b*) and for a geophysical flow in a sphere by Clark, Clark, Thomas & Lee (1971). The results are all qualitatively similar: on the time scale t_0, a quasi-steady state is reached in which the interior angular velocity is non-uniform in space. The final uniform rotation is reached only on the longer time scale L^2/ν. When the stratification is strong ($S \gg 1$), the spin-down is confined to layers of thickness $L/S^{\frac{1}{2}}$ adjacent to the top and bottom of the cylinder, and the time required to reach the quasi-steady state is of the order of $t_0 S^{-\frac{1}{2}}$.

In all of the above work, however, the Prandtl number

$$P = \nu/\kappa, \tag{4}$$

where κ is the thermal diffusivity, is of the order of unity. No one has yet explored the spin-down problem for the case with very small Prandtl number in spite of its astronomical importance. Because of efficient radiative transfer, the Prandtl number in stellar interiors is almost always small. In particular, this is the case for the solar interior.

In this paper, we discuss the laboratory spin-down problem for a Boussinesq fluid with very small Prandtl number. The fluid in a circular cylinder is initially in uniform rotation about the vertical axis of symmetry. The temperature near the top is held higher than that near the bottom to establish a stable stratification. Then the angular velocity of the cylinder is changed abruptly by a slight amount, while the horizontal walls of the cylinder are kept at their original temperatures and the vertical wall is kept at the original temperature or is thermally insulated. Our problem is to investigate the response of the fluid to this change.

As will be discussed in detail in § 3, the nature of the spin-down process depends on the ordering of the Prandtl number with respect to the Ekman number. For

$$P = O(E^a), \tag{5}$$

where $a > 0$, the spin-down process is divided into three different categories. If $a < \frac{1}{2}$, the thermal diffusion time

$$L^2/\kappa = t_0 O(E^{a-\frac{1}{2}}) \tag{6}$$

is much greater than the time t_0, and the process is qualitatively similar to that in the case with $P = O(1)$. For $a > \frac{1}{2}$, the diffusion time is much shorter than the time t_0, and the temperature field is determined entirely by the thermal boundary conditions, with the consequence that the buoyancy force is unimportant in the spin-down process. Thus this case is similar to that of a homogeneous fluid.

Finally, the case with $a = \frac{1}{2}$ gives us a qualitatively new spin-down process which is analyzed in detail in the present paper.

2. Basic equations

The equations used here are those of the Boussinesq approximation, linearized about a basic state of uniform rotation and constant temperature gradient (see, for example, Barcilon & Pedlosky 1967*a*). The angular velocity Ω and the temperature gradient β are in the positive Z direction and the gravity g is in the negative Z direction. The container is a circular cylinder of radius r_0 and height $2Hr_0$. We assume an axisymmetric motion and use a rotating cylindrical coordinate system (r, θ, Z). The equations are made dimensionless by taking r_0 for the length scale, the homogeneous fluid spin-down time $t_0 = E^{-\frac{1}{2}}\Omega^{-1}$ for the time scale, $r_0\Omega$ for the azimuthal velocity scale, $E^{\frac{1}{2}}r_0\Omega$ for the meridional velocity scale and $r_0\Omega^2/(\alpha g)$ for the temperature scale, where α is the coefficient of thermal expansion. Finally, the Prandtl number is expressed as follows with respect to the Ekman number:

$$P = \sigma E^a, \tag{7}$$

where σ is a constant of order unity. The dimensionless linearized equations are then

$$E\mathscr{L}\left\{E^{\frac{1}{2}}\mathscr{L} - \frac{\partial}{\partial t}\right\}\psi + 2\frac{\partial q_\theta}{\partial Z} = \frac{\partial T}{\partial r}, \tag{8}$$

$$\left\{E^{\frac{1}{2}}\mathscr{L} - \frac{\partial}{\partial t}\right\}q_\theta = 2\frac{\partial \psi}{\partial Z}, \tag{9}$$

and

$$\left(\sigma\frac{\partial}{\partial t} - E^{\frac{1}{2}-a}\Delta\right)T = \frac{\sigma S}{r}\frac{\partial(r\psi)}{\partial r}, \tag{10}$$

where

$$\mathscr{L} = \Delta - \frac{1}{r^2}, \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial Z^2}. \tag{11}$$

Here T is the perturbation temperature, and ψ is the stream function of the meridional current $\mathbf{q}$, with $\mathbf{q} = -\operatorname{curl}(\psi\mathbf{e}_\phi)$. In the above equation

$$S = \frac{\alpha\beta g}{\Omega^2} = \frac{N^2}{\Omega^2}, \quad E = \frac{\nu}{\Omega r_0^2}, \tag{12}$$

where N is the Brunt–Väisälä frequency.

The initial conditions are the statement that the fluid is in basic rotating equilibrium until a certain instant of time:

$$\psi = q_\theta = T = 0, \quad \text{for} \quad -H \leqslant Z \leqslant H \quad (0 \leqslant r \leqslant 1, t \leqslant 0). \tag{13}$$

The boundary conditions for $t > 0$ for the two cases that we consider are given below.

(i) Prescribed side-wall temperature:

$$\psi = \partial\psi/\partial Z = T = 0, \quad q_\theta = \omega r, \quad \text{for} \quad Z = \pm H \quad (0 \leqslant r \leqslant 1), \tag{14}$$

$$\psi = \partial\psi/\partial r = T = 0, \quad q_\theta = \omega, \quad \text{for} \quad r = 1 \quad (-H \leqslant Z \leqslant H). \tag{15}$$

(ii) Thermally insulated side wall:

$$\psi = \partial\psi/\partial Z = T = 0, \quad q_\theta = \omega r, \quad \text{for} \quad Z = \pm H \quad (0 \leqslant r \leqslant 1), \tag{16}$$

$$\psi = \partial\psi/\partial r = \partial T/\partial r = 0, \quad q_\theta = \omega, \quad \text{for} \quad r = 1 \quad (-H \leqslant Z \leqslant H). \tag{17}$$

3. Ordering of the Prandtl number

Before going ahead directly with the case of most interest, let us determine how the spin-down process depends on the ordering of the Prandtl number with respect to the Ekman number (i.e. the choice of exponent a in (7)). In the following, we will often have occasion to compare the present results with earlier work carried out for order unity Prandtl number. For brevity, we refer to the earlier work as the case $P = O(1)$.

For $a < \frac{1}{2}$, the interior equations (obtained by letting $E \to 0$ in (8)–(10)) are

$$2\partial q_\theta/\partial Z = \partial T/\partial r, \tag{18}$$

$$\partial q_\theta/\partial t = -2\,\partial\psi/\partial Z, \tag{19}$$

and

$$\partial T/\partial t = (S/r)\,\partial(r\psi)/\partial r. \tag{20}$$

The boundary conditions on the interior flow, which can be obtained by a detailed boundary-layer analysis similar to the analysis in appendix A, are as follows:

$$\mp 2\psi + q_\theta = \omega r, \quad \text{on} \quad Z = \pm H \quad \text{for cases (i) and (ii)}, \tag{21}$$

$$T = 0 \quad \text{on} \quad r = 1 \quad \text{for the case (i)}, \tag{22a}$$

and

$$\psi = 0 \quad \text{on} \quad r = 1 \quad \text{for the case (ii)}. \tag{22b}$$

We may compare the problem defined by (18)–(22) with Sakurai's (1969a,b) results for the case $P = O(1)$. For case (ii), Sakurai's equations and boundary conditions are identical with the present ones. For case (i), the only difference is the side-wall boundary condition. The condition obtained by Sakurai (1969a) is equivalent to

$$T + 2^{\frac{1}{2}}(PS)^{\frac{3}{4}}\,\psi = 0. \tag{23}$$

Thus for $a < \frac{1}{2}$, the spin-down process for case (i) is similar to (but not identical with) that for $P = O(1)$, and for case (ii) is identical with that for $P = O(1)$.

For $a > \frac{1}{2}$, the interior equations are (18), (19) and

$$\Delta T = 0. \tag{24}$$

The boundary conditions on these, which again are derived by a boundary-layer analysis, are

$$\mp 2\psi + q_\theta = \omega r \quad \text{and} \quad T = 0, \quad \text{on} \quad Z = \pm H \quad \text{for cases (i) and (ii)}, \tag{25}$$

and

$$T = 0 \quad \text{on} \quad r = 1 \quad \text{for case (i)}, \tag{26a}$$

$$\partial T/\partial r = 0 \quad \text{on} \quad r = 1 \quad \text{for case (ii)}. \tag{26b}$$

The solution of (24) subject to the conditions (25) and (26) is $T = 0$. Then (18), (19) and the remaining boundary conditions reduce to those for a homogeneous fluid. Thus for $a > \frac{1}{2}$, the spin-down process is identical to that for a homogeneous fluid.

Since the cases $a < \frac{1}{2}$ and $a > \frac{1}{2}$ give us nothing new, while the case with $a = \frac{1}{2}$ is distinguished because of the fact that the thermal diffusion term ΔT is of the same order of magnitude as the other interior terms, we hereafter restrict ourselves to the case $a = \frac{1}{2}$.

The basic equations for the interior inviscid flow, for $a = \frac{1}{2}$ are

$$2\,\partial q_\theta/\partial Z = \partial T/\partial r, \tag{27}$$

$$-\partial q_\theta/\partial t = 2\,\partial\psi/\partial Z, \tag{28}$$

and

$$\left(\sigma\frac{\partial}{\partial t} - \Delta\right) T = \frac{\sigma S}{r}\frac{\partial}{\partial r}(r\psi). \tag{29}$$

The boundary conditions for the interior flow are derived in appendix A by a boundary-layer analysis of the full equations (see also Barcilon & Pedlosky 1967*b*). The resulting conditions on the interior flow are

$$\mp 2\psi + q_\theta = \omega r, \quad T = 0, \quad \text{on} \quad Z = \pm H \quad \text{for cases (i) and (ii)}, \tag{30}$$

$$T = 0 \quad \text{on} \quad r = 1 \quad \text{for case (i)}, \tag{31a}$$

$$\partial T/\partial r + \sigma S\psi = 0 \quad \text{on} \quad r = 1 \quad \text{for case (ii)}. \tag{31b}$$

The first of equations (30) describes the pumping of the meridional current by the disturbance of the geostrophic balance in the Ekman layer. From the second of (30), we see that there is no horizontal thermal boundary layer, in contrast with the case $P = O(1)$. The condition (31*a*) again corresponds to the absence of a thermal boundary layer (in lowest order), while (31*b*) is essentially the condition that the total energy flux (convective plus conductive) vanish at the side wall.

The initial conditions are

$$T = q_\theta = 0 \quad \text{for} \quad t = 0. \tag{32}$$

As discussed by Greenspan & Howard (1963), the initial condition on ψ must be dropped since meridional currents of order unity are built up during the establishment of the Ekman layer (within the time scale Ω^{-1}).

It is of interest that the equations (27)–(29), with the boundary conditions (30) and (31), have an energy-dissipation integral for both the cases (i) and (ii):

$$\frac{d}{dt}\iiint_V \{\tfrac{1}{2}\sigma T^2 + \tfrac{1}{2}S\sigma(q_\theta - \omega r)^2\}\,dV$$
$$= -\iiint_V (\operatorname{grad} T)^2\,dV - \iint_{Z=H} S\sigma(q_\theta - \omega r)^2\,ds - \iint_{Z=-H} S\sigma(q_\theta - \omega r)^2\,ds. \tag{33}$$

Here V is the total volume of the cylinder. The important conclusions derived from the above are: (*a*) the solution of the initial-value problem for the interior flow equations is unique; (*b*) the asymptotic state to which our interior flow approaches is rigid-body rotation with the new angular velocity of the container ($q_\theta \equiv \omega r$, $T \equiv 0$); (*c*) the negative-definite driving terms on the right-hand side are associated with thermal relaxation throughout the volume (the $(\operatorname{grad} T)^2$ term) and the Ekman layers (the $(q_\theta - \omega r)^2$ terms), but not the side-wall layers. Thus the side-wall layers are passive in the spin-down process. Since the quantities in (33) are all of order unity with respect to the Ekman number, the time scale for approaching the asymptotic state is of order unity (that is, of the order of the homogeneous spin-down time $t_0 = \Omega^{-1}E^{-\frac{1}{2}}$). The above energy integral is easily extended to the case of an arbitrary axisymmetric container with arbitrary time variation of the wall velocity (appendix B). This generalization of the energy integral shows that the asymptotic state is rigid-body rotation for any axisymmetric container. This is in strong contrast to the case $P = O(1)$, where the asymptotic state is one of non-uniform rotation. In this respect, the present case is more nearly like that of a homogeneous fluid. In other respects, however, our situation is definitely different from that of a homogeneous fluid. For example, the transient spin-down is non-uniform in Z – there is no Taylor–Proudman column as in the homogeneous case, as is evident from equations (27) and (28).

4. Solution for the interior flow

Case (i): *prescribed side-wall temperature*

We seek a solution in the following form:

$$T = \sum_{n=1}^{\infty} T_n(Z,t)\, J_0(\omega_n r), \tag{34}$$

$$q_\theta = \sum_{n=1}^{\infty} q_n(Z,t)\, J_1(\omega_n r), \tag{35}$$

and

$$\psi = \sum_{n=1}^{\infty} \psi_n(Z,t)\, J_1(\omega_n r), \tag{36}$$

where J_0 and J_1 are Bessel functions and ω_n is the nth positive zero of J_0. Properties of such expansions are discussed briefly in appendix C. The series for T may be differentiated termwise twice, and the series for ψ may be differentiated termwise once with respect to r. Substitution of the series into equations (27)–(29) yields the following set of equations:

$$2(\partial q_n/\partial Z) = -\omega_n T_n, \tag{37}$$

$$\partial q_n/\partial t = -2\,\partial \psi_n/\partial Z, \tag{38}$$

and

$$\sigma \frac{\partial T_n}{\partial t} - \left(\frac{\partial^2}{\partial Z^2} - \omega_n^2\right) T_n = \sigma S \omega_n \psi_n. \tag{39}$$

The initial conditions are $\quad T_n(Z,0) = q_n(Z,0) = 0. \qquad (40)$

The boundary conditions on $Z = \pm H$ are readily applied with the aid of the following expansion:

$$r = \sum_{n=1}^{\infty} \mu_n J_1(\omega_n r), \tag{41}$$

where

$$\mu_n = \frac{\displaystyle\int_0^1 r^2 J_1(\omega_n r)\, dr}{\displaystyle\int_0^1 r\{J_1(\omega_n r)\}^2\, dr} = \frac{4}{\omega_n^2 J_1(\omega_n)}. \tag{42}$$

These conditions are

$$\mp 2\psi_n + q_n = \mu_n \omega, \quad T_n = 0, \quad \text{on} \quad Z = \pm H. \tag{43}$$

The above initial-value problem can be solved by the Laplace transform in time. We denote the transform variable by τ, the transforms of q_θ, ψ and T by Q, Ψ and $\mathscr{T}$, respectively, and we write $D = d/dZ$. Then the transformed equations and boundary conditions are

$$2DQ_n = -\omega_n \mathscr{T}_n, \tag{44}$$

$$\tau Q_n = -2D\Psi_n, \tag{45}$$

and
$$\sigma\tau\mathscr{T}_n - (D^2 - \omega_n^2)\mathscr{T}_n = \sigma S \omega_n \Psi_n, \tag{46}$$

with
$$\mp 2\Psi_n + Q_n = \mu_n \omega/\tau, \quad \mathscr{T}_n = 0, \quad \text{on} \quad Z = \pm H. \tag{47}$$

The solution of (44)–(47) is straightforward. The result is

$$Q_n = A_n \left\{ \frac{\cosh(\alpha_n Z)}{\alpha_n \sinh(\alpha_n H)} - \frac{\cosh(\beta_n Z)}{\beta_n \sinh(\beta_n H)} \right\}, \tag{48}$$

$$\mathscr{T}_n = \frac{2A_n}{\omega_n} \left\{ \frac{\sinh(\beta_n Z)}{\sinh(\beta_n H)} - \frac{\sinh(\alpha_n Z)}{\sinh(\alpha_n H)} \right\}, \tag{49}$$

and
$$\Psi_n = \frac{\tau A_n}{2} \left\{ \frac{\sinh(\beta_n Z)}{\beta_n^2 \sinh(\beta_n H)} - \frac{\sinh(\alpha_n Z)}{\alpha_n^2 \sinh(\alpha_n H)} \right\}, \tag{50}$$

where
$$A_n = \mu_n \omega/\tau D_n, \tag{51}$$

and
$$D_n = \frac{\coth(\alpha_n H)}{\alpha_n} - \frac{\coth(\beta_n H)}{\beta_n} + \tau \frac{\beta_n^2 - \alpha_n^2}{\alpha_n^2 \beta_n^2}, \tag{52}$$

with
$$\begin{Bmatrix} \alpha_n^2 \\ \beta_n^2 \end{Bmatrix} = \tfrac{1}{2}(\sigma\tau + \omega_n^2) \mp \tfrac{1}{2}\{(\sigma\tau + \omega_n^2)^2 - \sigma S \omega_n^2 \tau\}^{\frac{1}{2}}. \tag{53}$$

The functions Q_n, $\mathscr{T}_n$ and Ψ_n are even functions of α_n and β_n, and are symmetric with respect to the exchange of α_n and β_n. It follows, therefore, that these functions do not have branch points as functions of τ, even though α_n and β_n have. Another point to be noted is that, although $\alpha_n = \beta_n$ gives us a root of the denominator D_n for which the real part of τ is positive, the numerators for Q_n, $\mathscr{T}_n$ and Ψ_n also vanish which shows that such points are removable singularities. The only remaining singularities are poles at $\tau = 0$ and in the left half plane of τ. It is easy to verify that the solutions satisfy the initial conditions (40). The asymptotic state for $t \to \infty$ also may be verified. The residue of (48) at $\tau = 0$ is $\mu_n \omega$ (by (51) to (53)), which, by (41) and (42) leads immediately to the fact that the asymptotic state is one of rigid-body rotation. Similar estimates lead to the conclusions that T and ψ tend to zero in the asymptotic state.

Since the initial and the asymptotic states are described, the remaining problem is the description of the time variation connecting these two limiting states. As a single quantity which describes the spin-down, we investigate the total angular momentum $\mathscr{J}$, normalized to unity for the final state of the rigid-body rotation. The Laplace transform of $\mathscr{J}$ is as follows:

$$\mathscr{J} = \frac{2}{H\omega} \int_{-H}^{H} dZ \int_0^1 r^2 Q \, dr, \tag{54}$$

$$= \frac{32}{H\tau} \sum_{n=1}^{\infty} \frac{\beta_n^2 - \alpha_n^2}{\omega_n^4 D_n \alpha_n^2 \beta_n^2}. \tag{55}$$

The residue of the above at $\tau = 0$ is equal to

$$32 \sum_{n=1}^{\infty} 1/\omega_n^4.$$

It is easy to prove that the above is equal to unity, by Parseval's theorem applied to the series (41), and this verifies that the asymptotic state is rigid-body rotation.

Case (ii): *thermally insulated side wall*

For the case of a thermally insulated side wall, the appropriate sets of functions to use are $\{1, J_0(\gamma_n r)\}$ and $\{J_1(\gamma_n r)\}$, where γ_n is the nth positive root of J_1. Each of these sets is complete and orthogonal with respect to r on $[0, 1]$ (see appendix C). We seek a solution of the following form:

$$T = T_0(Z,t) + \sum_{n=1}^{\infty} T_n(Z,t)\, J_0(\gamma_n r), \tag{56}$$

$$q_\theta = \sum_{n=1}^{\infty} q_n(Z,t)\, J_1(\gamma_n r), \tag{57}$$

and

$$\psi = \sum_{n=1}^{\infty} \psi_n(Z,t)\, J_1(\gamma_n r). \tag{58}$$

Since q_θ and ψ do not vanish on $r = 1$, while the functions $J_1(\gamma_n r)$ do vanish there, the convergence of the series (57) and (58) is poor (like $1/n$), and these series may not be differentiated termwise with respect to r. The series (56) for T, on the other hand, may be differentiated termwise once (see appendix C). Substitution of the above into (27) and (28) yields the following two equations:

$$2(\partial q_n/\partial Z) = -\gamma_n T_n, \tag{59}$$

and

$$\partial q_n/\partial t = -2(\partial \psi_n/\partial Z). \tag{60}$$

The third equation is rewritten as

$$\sigma\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial Z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(\sigma S\psi + \frac{\partial T}{\partial r}\right)\right\}. \tag{61}$$

The quantity on the right-hand side has the expansion

$$\sigma S\psi + \partial T/\partial r = \sum_{n=1}^{\infty} (\sigma S\psi_n - \gamma_n T_n)\, J_1(\gamma_n r). \tag{62}$$

Since $\sigma S\psi + \partial T/\partial r$ vanishes on $r = 1$ (by the boundary condition (31*b*)), the series (62) may be differentiated termwise with respect to r. Substitution of the above into (61) yields

$$\sigma\,\partial T_0/\partial t = \partial^2 T_0/\partial Z^2 \tag{63}$$

and

$$\sigma\,\partial T_n/\partial t - \partial^2 T_n/\partial Z^2 = \gamma_n(\sigma S\psi_n - \gamma_n T_n) \quad (n \geqslant 1). \tag{64}$$

Equations (59), (60), (63) and (64) are to be solved under the following initial and boundary conditions:

$$T_n(Z,0) = 0, \quad q_n(Z,0) = 0, \tag{65}$$

and

$$\mp 2\psi_n + q_n = \rho_n \omega, \quad T_n = 0 \quad \text{on} \quad Z = \pm H, \tag{66}$$

where
$$r = \sum_{n=1}^{\infty} \rho_n J_1(\gamma_n r), \tag{67}$$

with
$$\rho_n = \frac{\int_0^1 r^2 J_1(\gamma_n r)\,dr}{\int_0^1 r\{J_1(\gamma_n r)\}^2\,dr} = \frac{2}{\gamma_n J_2(\gamma_n)}. \tag{68}$$

It is easily shown that $T_0 = 0$. The form of the remaining problem is identical to that of case (i). Thus the Laplace transforms of the solutions are given by (48) through (53) with the following changes:

$$\omega_n \to \gamma_n, \quad \mu_n \to \rho_n. \tag{69}$$

The angular momentum is

$$\mathscr{J} = \frac{2}{H\omega}\int_{-H}^{H} dZ \int_0^1 r^2 Q\,dr \tag{70}$$

$$= \frac{8}{H\tau}\sum_{n=1}^{\infty}\frac{\beta_n^2-\alpha_n^2}{\gamma_n^2 D_n \alpha_n^2 \beta_n^2}. \tag{71}$$

Similar considerations as in case (i) show that the solution satisfies the initial condition (32), and that the asymptotic state as $t \to \infty$ is one of rigid-body rotation ($q_\theta = \omega r$) with vanishing values of ψ and T, and with $J \to 1$.

We show now that a closed form expression may be obtained for the Laplace transform of the azimuthal velocity on the side wall ($r \to 1$). Consider the series for Q. We have

$$Q(r, Z, \tau) = \sum_{n=1}^{\infty} Q_n(Z, \tau)\, J_1(\gamma_n r), \tag{72}$$

with
$$Q_n = \frac{\rho_n \omega}{D_n \tau}\left\{\frac{\cosh(\alpha_n Z)}{\alpha_n \sinh(\alpha_n H)} - \frac{\cosh(\beta_n Z)}{\beta_n \sinh(\beta_n H)}\right\}. \tag{73}$$

For large n, we have

$$\beta_n^2 = \gamma_n^2 + O(1), \quad \alpha_n^2 = \tfrac{1}{4}\sigma S\tau\{1 + O(\gamma_n^{-2})\}, \tag{74}$$

and
$$D_n = \frac{\coth[\frac{1}{2}H(\sigma S\tau)^{\frac{1}{2}}]}{\frac{1}{2}H(\sigma S\tau)^{\frac{1}{2}}} + \frac{4}{\sigma S} + O(\gamma_n^{-1}). \tag{75}$$

Thus
$$Q_n = \rho_n F(Z, \tau)\{1 + O(\gamma_n^{-1})\}, \tag{76}$$

where
$$F(Z, \tau) = \frac{\omega \cosh[\frac{1}{2}Z(\sigma S\tau)^{\frac{1}{2}}]}{\tau\{2(\tau/\sigma S)^{\frac{1}{2}}\sinh[\frac{1}{2}H(\sigma S\tau)^{\frac{1}{2}}] + \cosh[\frac{1}{2}H(\sigma S\tau)^{\frac{1}{2}}]\}}. \tag{77}$$

Finally, we have
$$\sum_{n=1}^{\infty} \rho_n J_1(\gamma_n r) = r \to 1 \quad \text{as} \quad r \to 1, \tag{78}$$

and
$$\sum_{n=1}^{\infty} \rho_n O(\gamma_n^{-1})\, J_1(\gamma_n r) \to 0 \quad \text{as} \quad r \to 1. \tag{79}$$

The last limit above comes from the fact that $\rho_n O(\gamma_n^{-1}) = O(\gamma_n^{-\frac{3}{2}})$ implying the uniform convergence, and hence the continuity of the sum at the point $r = 1$. Thus we get the following result for the Laplace transform of the side-wall azimuthal velocity:

$$Q(1, Z, \tau) = F(Z, \tau). \tag{80}$$

It is interesting to note that the above side-wall distribution of the azimuthal velocity can be obtained more directly. If we let $q_\theta(1, Z, t) = f(Z, t)$, then the following equation is obtained from (27), (28) and (31*b*):

$$\frac{\partial f}{\partial t} = \frac{2}{\sigma S}\frac{\partial^2 T}{\partial r\,\partial Z} = \frac{4}{\sigma S}\frac{\partial^2 f}{\partial Z^2}. \tag{81}$$

From equations (27) and (31*b*), we get

$$2\,\partial f/\partial Z = -\sigma S \psi(1, Z, t). \tag{82}$$

If we let $Z \to \pm H$, and use (30), along with the fact that q_θ and ψ are continuous, then we obtain the following end-point conditions on (81):

$$f \pm (4/\sigma S)\,\partial f/\partial Z = \omega, \quad \text{at} \quad Z = \pm H. \tag{83}$$

It is easily shown that the Laplace transform of the above f is just $F(Z, \tau)$. It is worth noting that this kind of manipulation of equations and boundary conditions near the corners must be done with care, since some quantities, $\partial T/\partial r$ for example, are not continuous there.

5. Results and discussion

The important quantities in the Laplace inversion are the singularities of the Laplace transforms (48)–(50). In this case the singularities consist of a pole at $\tau = 0$, which gives the asymptotic state with $q_\theta = \omega r$, $\psi = 0$ and $T = 0$, and poles in the left half τ plane, which give information about the transient approach to the asymptotic state.

The poles in the left half plane are the zeros of D_n. If we let

$$\tau = -\lambda^2/(\Gamma H), \tag{84}$$

for λ real and positive,

$$\Gamma = \tfrac{1}{4}\sigma S H, \tag{85}$$

then the equation $D_n = 0$ can be put into the following form:

$$\frac{\cot X_n}{X_n} = \frac{X_n^2 + Y_n^2}{\Gamma H^2 \omega_n^2} - \frac{\coth Y_n}{Y_n}, \tag{86}$$

where

$$X_n = (M_n - N_n)^{\frac{1}{2}}, \tag{87}$$

$$Y_n = (M_n + N_n)^{\frac{1}{2}}, \tag{88}$$

$$M_n = (N_n^2 + H^2\omega_n^2\lambda^2)^{\frac{1}{2}}, \tag{89}$$

and

$$N_n = \tfrac{1}{2}H^2\omega_n^2 - (2\lambda^2/S). \tag{90}$$

In any interval $(m-1)\pi < X_n < m\pi$ the function $(\cot X_n)/X_n$ varies from $+\infty$ to $-\infty$, while the right-hand side of (86) is bounded. Thus there is a root λ_{nm} in every such interval. Since this is true for each n (i.e. for each term in the radial expansion), there are infinitely many λ_{nm} in the interval $(m-1)\pi < X_n < m\pi$. Since $X_n \simeq \lambda$ for large n, we have the result that the roots λ_{nm} (for each fixed m) have a limit point in the interval $(m-1)\pi < \lambda < m\pi$. As simple measures of these roots, we may take the limiting values

$$\beta_m = \lim_{n\to\infty} \lambda_{nm}. \tag{91}$$

From (86), one can show that β_m is the mth root of

$$\tan\beta = \Gamma/\beta. \tag{92}$$

The smallest root of this equation is a measure of the spin-down time, since the solution behaves like $\exp(-t/t_c)$, where

$$t_c = \Gamma H/\beta_1^2 = t_0(\Gamma/\beta_1^2). \tag{93}$$

For weak stratification ($\Gamma \to 0$) it follows from (92) that $\beta_1 \to \Gamma^{\frac{1}{2}}$ so that $t_c \to t_0$, which is the known spin-down time for a homogeneous fluid. For strong stratification ($\Gamma \to \infty$), $\beta_1 \to \frac{1}{2}\pi$ so that $t_c \to (4\Gamma/\pi^2)\,t_0$, which becomes indefinitely large with Γ. The dimensional spin-down time $\tilde{t}_c$ in this limit ($\Gamma \to \infty$) is particularly simple:

$$\tilde{t}_c = \Omega^{-1}E^{-\frac{1}{2}}t_c = S(h^2/\pi^2\kappa). \tag{94}$$

Thus the spin-down time for strong stratification is simply the thermal diffusion time increased by the factor of the stratification number.

The above analysis is based on the identification of β_1 as representative of the first root. A more accurate treatment is also of interest. We have calculated the exact first zero of D_n from equation (86) for $\sigma = 1$ and $H = 1$ for a large number of S values, both for fixed temperature and insulated ($\omega_n \to \gamma_n$) side walls. The results are shown in figure 1, where $\log t_c$ is plotted as a function of $\log S$. For comparison, the limit-point root obtained from equation (92) is also plotted. The three roots are nearly the same for small stratification. For larger S, the limit-point root gives a value too large, and the exact roots show that the spin-down process is slower for insulated side walls. This is in complete qualitative agreement with the results of Sakurai (1969*a*, *b*) for the case $P = O(1)$.

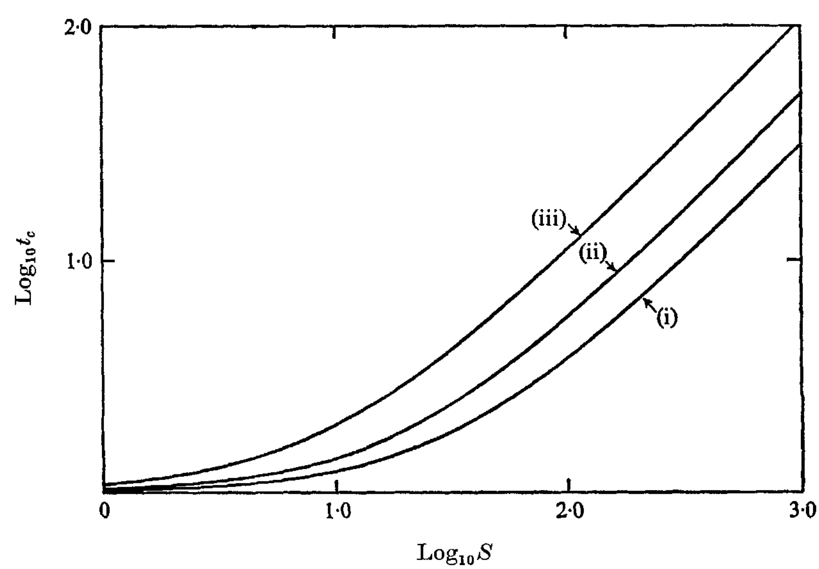

FIGURE 1. Spin-down time t_c as a function of the stratification parameter S for fixed side-wall temperature (i), for an insulated side wall (ii), and approximate limit-point value (iii). The other parameter values are $H = 1$ and $\sigma = 1$.

We consider now a more elaborate calculation – namely the calculation of the total angular momentum J as a function of time. Although the method to be used will work for a wide range of parameter values, we have chosen $S = 4$, $\sigma = 1$ and $H = 1$ since there are some simplifications in this case. We present numerical results for both boundary conditions but since the calculations are very similar in the two cases, we give the details only for the case of fixed side-wall temperature.

For our choice of parameters, the Laplace transform (55) becomes

$$\mathscr{J} = \frac{32}{\tau^2} \sum_{n=1}^{\infty} \frac{(\tau - \omega_n^2)}{\omega_n^6 D_n}. \tag{95}$$

From the residue theorem and previous considerations on the roots of $D_n = 0$, we derive the following expression for $J(t)$:

$$J(t) = 1 - \sum_{n=1}^{\infty} L_m(t), \tag{96}$$

where

$$L_m(t) = \sum_{n=1}^{\infty} C_{nm} \exp(-\lambda_{nm}^2 t), \tag{97}$$

with

$$C_{nm} = 32(\lambda_{nm} + \omega_n^2) \Big/ \lambda_{mn}^2 \omega_n^4 \left(\lambda_{nm}^2 + \tfrac{1}{2}\omega_n^2 \left[\frac{1}{\sin^2 \lambda_{nm}} + \frac{\cot \lambda_{nm}}{\lambda_{nm}}\right]\right). \tag{98}$$

For large n

$$\lambda_{nm} \sim \beta_m, \quad C_{nm} \sim \frac{32}{\omega_n^4(1 + \frac{1}{2}\beta_m^2)}, \tag{99}$$

where β_m is the mth root of (92) with $\Gamma = 1$. The convergence in the series (97) is thus like n^{-4}. We can improve the convergence by adding and subtracting asymptotic expressions to the terms in (97) to get

$$\begin{aligned} L_m(t) &= \frac{32 \exp(-\beta_m^2 t)}{1 + \frac{1}{2}\beta_m^2} \sum_{n=1}^{\infty} \frac{1}{\omega_n^4} + \sum_{n=1}^{\infty} E_{nm} \\ &= \frac{\exp(-\beta_m^2 t)}{1 + \frac{1}{2}\beta_m^2} + \sum_{n=1}^{\infty} E_{nm}, \end{aligned} \tag{100}$$

where

$$E_{nm} = C_{nm} \exp(-\lambda_{nm}^2 t) - \frac{32 \exp(-\beta_m^2 t)}{1 + \frac{1}{2}\beta_m^2}. \tag{101}$$

One can show that the terms E_{nm} decrease like n^{-6}, so that only a very few terms need to be kept. Once the L_m's are calculated, $J(t)$ may be calculated from (96). The roots β_m increase rapidly with m so that, again, only a few terms are needed (unless t is very small). The calculations are similar for the case of an insulated side wall. The principal difference is that the series for L_m, corresponding to (97), converges only like n^{-2}. The transformations corresponding to (99)–(101) give a series with n^{-4} convergence. Figure 2 shows a plot of $J(t)$ for both boundary conditions. Again we see that the spin-down is slower for the insulated wall, although the difference is not nearly so great as in the $P = O(1)$ case (Sakurai 1969*a*, *b*).

As another example of the behaviour of the flow field, we consider the interior azimuthal velocity at $r = 1$ for the case of an insulated side wall. The Laplace transform of this function is given by (77). The inversion is straight forward, and the result is

$$\frac{q_\theta}{\omega} = 1 - \sum_{n=1}^{\infty} \frac{2\cos(\beta_n y)\exp(-\beta_n^2 \hat{t})}{\cos(\beta_n)(1+\Gamma+\beta_n^2/\Gamma)}, \tag{102}$$

where β_n is the nth root of equation (92), and

$$y = Z/H, \quad \hat{t} = t/(\Gamma H). \tag{103}$$

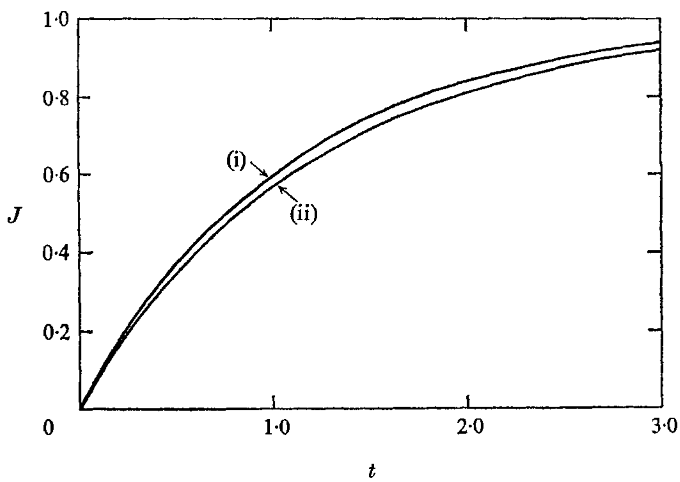

FIGURE 2. Angular momentum J as a function of time t for fixed side-wall temperature (i), and insulated side wall (ii). The parameter values are $S = 4$, $H = 1$ and $\sigma = 1$.

It is interesting to note that the time-decay exponents are simply those corresponding to the limit-point roots. In the limit $\Gamma \to \infty$, the formula (102) becomes

$$q_\theta/\omega = 1 - (2/\pi)\sum_{n=1}^{\infty}(-1)^{n+1}(n-\tfrac{1}{2})^{-1}\cos[(n-\tfrac{1}{2})\pi y]\exp[-(n-\tfrac{1}{2})^2\pi^2\hat{t}]. \tag{104}$$

Figure 3 shows q_θ/ω as a function of y, for various values of $\hat{t}$, for $\Gamma = 1$, 10, 100 and ∞. The graphs show clearly the way in which the spin-down process becomes more non-uniform in height as the stratification increases. These results are similar to those obtained by Sakurai (1970) for the case $P = O(1)$.

Now we consider briefly the physical significance of the ordering $P = O(E^{\frac{1}{2}})$ on which the present analysis is based. One characteristic of this ordering (as discussed in § 1 above) is the fact that the thermal diffusion time (L^2/κ) is comparable to the spin-down time $t_0 = \Omega^{-1}E^{-\frac{1}{2}}$. Further insight is obtained from a consideration of circulation velocities. From the Ekman layer analysis, one obtains a dimensional circulation velocity $V_E \sim \omega L\Omega E^{\frac{1}{2}}$, where ω is the Rossby number as in the above analysis. Another characteristic circulation velocity may be obtained from an order-of-magnitude analysis of the interior equations, including the effects of thermal diffusion. One finds an interior circulation velocity $V_I \sim \omega\kappa(LS)^{-1}$, provided that $S \gtrsim 1$. (This velocity is entirely analogous to the Eddington–Sweet velocity in the theory of meridional circulations in stars.) These two velocities are comparable if $P \sim S^{-1}E^{\frac{1}{2}}$, in particular if $S = O(1)$ and

$P = O(E^{\frac{1}{2}})$. Thus the case $P = O(E^{\frac{1}{2}})$ corresponds to the situation in which the circulations allowed by the interior dynamics are comparable with those produced by the Ekman pumping mechanism.

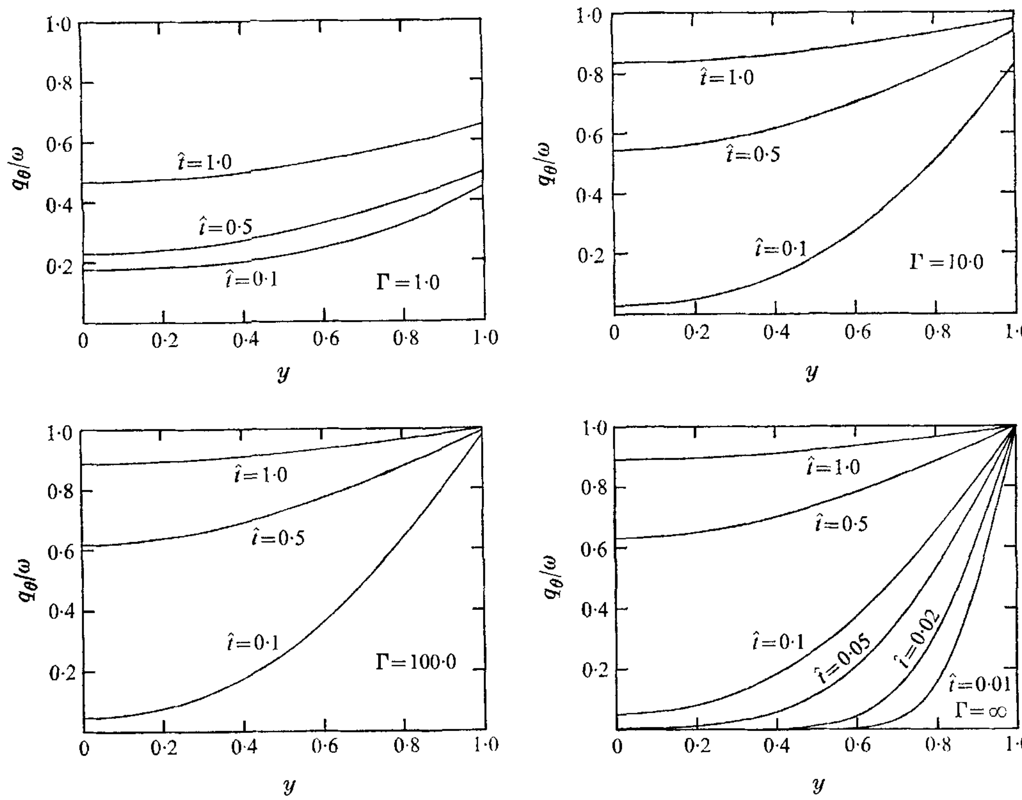

FIGURE 3. Interior azimuthal velocity (q_θ/ω) on the side wall as a function of height y for various stratification parameters Γ and various times $\hat{t}$. The centre plane of the cylinder is $y = 0$ and the top is $y = 1$. The side wall is insulated.

6. Solar spin-down problem

The suggestion by Dicke (1964) and Roxburgh (1964) that the interior of the sun may be rapidly rotating has led to considerable controversy. The possible importance of Ekman spin-down was first pointed out by Howard, Moore & Spiegel (1967). They estimated that spin-down would destroy the differential rotation proposed by Dicke in a time of the order of 10^9 years, close enough to the age of the sun to indicate that a quantitative treatment is necessary for a decisive answer. An extensive review of the whole solar rotation problem has been given recently by Dicke (1970). The present authors' points of view have been given in detail elsewhere (Sakurai 1970; Clark, Thomas & Clark 1969). Our purpose here is to draw what further conclusions we can from the present work.

Consider first the parameter range most relevant for the solar spin-down problem, $P = O(E^{\frac{1}{2}})$ and $S \gg 1$ (but $S = O(1)$ with respect to E). Then $V_I \ll V_E$ and a question of compatibility arises. By a lengthy but straightforward analysis of the solution given above, one may show the following. Apart from initial non-uniformities of short duration, the circulation velocity is of the order of V_I. Stratification reduces the flow through the Ekman layer to the point where it is compatible with the interior flow. A detailed discussion of these points will be given elsewhere.

To apply our above results to the solar spin-down problem, we use the following rough geometrical correspondence: the top and bottom of the cylinder ($Z = \pm H$) correspond to the north and south hemispheres of the lower boundary of the convection zone. The cylinder side wall corresponds to the equatorial plane, and the plane $Z = 0$ in the cylinder corresponds to the centre of the sun. The solar spin-down is caused by the continuous slowing down of the convection zone by the solar wind torque, and we may regard this as a succession of infinitesimal impulses. We use the present solution to study the penetration of each impulse, and we limit attention to the azimuthal velocity on the side wall (corresponding to the solar equatorial plane). Since the stratification is strong, we may use the formula (104). The quantity $\hat{t}$ in (104) may be expressed in terms of the dimensional time $\tilde{t}$ as follows:

$$\hat{t} = \frac{t}{\Gamma H} = \frac{4\Omega E^{\frac{1}{2}}}{\sigma S H^2}\tilde{t} = \left(\frac{2\Omega}{N}\right)^2 \frac{\kappa}{h^2}\tilde{t}. \tag{105}$$

Here $\tilde{t}$ is the time elapsed since the impulse at the boundary. For our present purposes, $\tilde{t}$ is the age of the sun ($\tilde{t} = 5 \times 10^9$ years) and $h = 0{\cdot}86R_0 = 6 \times 10^{10}$ cm is the inner radius of the convection zone. The quantities κ and N vary with position, and we have arbitrarily evaluated them at $0{\cdot}7R_0$ in Weymann's improved solar model, as tabulated by Schwarzschild (1958). The numbers are $\kappa = 3{\cdot}5 \times 10^7\,\mathrm{cm}^2/\mathrm{sec}$, $N = 1{\cdot}5 \times 10^{-3}\,\mathrm{sec}^{-1}$. Finally, there is the choice of Ω. Dicke (1970) has suggested a value of $\Omega_c = 5{\cdot}74 \times 10^{-5}\,\mathrm{sec}^{-1}$ for the core angular velocity, whereas the average surface value is $\Omega_S = 2{\cdot}87 \times 10^{-6}\,\mathrm{sec}^{-1}$. We take these two values as extremes in making our qualitative estimates. For $\Omega = \Omega_c$, we get $\hat{t} = 7{\cdot}5$ which implies $q_\theta/\omega = 1$ throughout. Thus any impulses applied to the rapidly rotating sun will penetrate the entire core. As spin-down proceeds, however, Ω becomes smaller, and the process becomes less efficient. For the slow extreme, $\Omega = \Omega_s$, we get $\hat{t} = 0{\cdot}019$, and the corresponding q_θ/ω is shown in figure 3 (for $\Gamma = \infty$ and $\hat{t} = 0{\cdot}02$). It is clear that impulses applied to the slowly rotating sun do not penetrate the core during the solar lifetime. In fact, for $\hat{t} = 0{\cdot}02$, the amplitude of q_θ/ω has dropped to $0{\cdot}1$ for $\hat{y} = 0{\cdot}7$, corresponding to a position $0{\cdot}6R_0$ in the sun. These results suggest that the angular velocity in the interior of the sun may be non-uniform. On the basis of our linear calculations, we cannot say whether non-uniformities as large as those required by Dicke are possible. Quantitative conclusions must await the analysis of the non-linear spin-down process in a sphere, for a compressible fluid of very small Prandtl number.

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Appendix A. Boundary-layer analysis

We give a brief summary of the boundary-layer analysis which leads to the boundary conditions (30) and (31) on the interior flow.

The basic equations are (8)–(10) with $a = \frac{1}{2}$ and the boundary conditions are (14)–(17). A scaling analysis gives the following results: there are horizontal layers of thickness $E^{\frac{1}{2}}$, and vertical layers of thickness $E^{\frac{1}{4}}$ and $E^{\frac{3}{8}}$. These results also follow directly from the work of Barcilon & Pedlosky (1967*b*).

The $E^{\frac{1}{2}}$ horizontal layer turns out to be an ordinary Ekman layer in which the corrections to q_θ and ψ are $O(1)$ while the first non-zero correction to T is $O(E)$. Since the analysis is essentially the same as the well-known ordinary Ekman layer analysis, we omit the details. The final results are the following conditions on the interior flow:

$$\mp 2\psi + q_\theta = \omega r \quad \text{at} \quad Z = \pm H, \tag{A 1}$$

and

$$T = 0 \quad \text{at} \quad Z = \pm H. \tag{A 2}$$

The principal difference between this case and the case $P = O(1)$ is that in the present case, there is no thermal boundary layer – the thermal condition is imposed directly on the interior flow.

We consider the side-wall layers in somewhat more detail. The $E^{\frac{1}{4}}$ layer is a merging of the shear layer and the hydrostatic layer discussed by Barcilon & Pedlosky (1967*b*). A detailed analysis shows that in this layer the corrections to q_θ and ψ are $O(1)$, while the correction to T is $O(E^{\frac{1}{4}})$. The other side-wall layer, with thickness $E^{\frac{3}{8}}$, is the buoyancy layer discussed by Barcilon & Pedlosky (1967*b*). In this layer, the correction to ψ is $O(1)$, the correction to q_θ is $O(E^{\frac{1}{8}})$ and the correction to T is $O(E^{\frac{3}{8}})$.

In the quantitative analysis of the side-wall layers, we use two stretched radial co-ordinates

$$\alpha = (1-r)/E^{\frac{1}{4}}, \quad \beta = (1-r)/E^{\frac{3}{8}}. \tag{A 3}$$

Near the side wall, the solution has the form

$$\psi = \psi^{(I)}(r, Z, t) + \overline{\psi}(\alpha, Z, t) + \hat{\psi}(\beta, Z, t), \tag{A 4}$$

$$q_\theta = q^{(I)}(r, Z, t) + \overline{q}(\alpha, Z, t) + E^{\frac{1}{8}}\hat{q}(\beta, Z, t), \tag{A 5}$$

and

$$T = T^{(I)}(r, Z, t) + E^{\frac{1}{4}}\overline{T}(\alpha, Z, t) + E^{\frac{3}{8}}\hat{T}(\beta, Z, t). \tag{A 6}$$

The expressions (A 4)–(A 6) contain only the lowest-order terms of each type, which is sufficient for our purposes here. The boundary-layer corrections are supposed to be exponentially small outside of their boundary layers.

We consider now the equations obtained by substituting (A 4)–(A 6) into the full equations. The functions $\psi^{(I)}$, $q^{(I)}$ and $T^{(I)}$ satisfy the interior equations (27)–(29). The $E^{\frac{1}{4}}$ layer equations are

$$\partial\overline{T}/\partial\alpha = -2\,\partial\overline{q}/\partial Z, \tag{A 7}$$

$$\partial^2\overline{q}/\partial\alpha^2 - \partial\overline{q}/\partial t = 2\,\partial\overline{\psi}/\partial Z \tag{A 8}$$

and

$$\partial^2\overline{T}/\partial\alpha^2 = \sigma S\,\partial\overline{\psi}/\partial\alpha. \tag{A 9}$$

The $E^{\frac{1}{3}}$ layer equations are

$$\partial\hat{T}/\partial\beta = -\partial^4\hat{\psi}/\partial\beta^4, \tag{A 10}$$

$$\partial^2\hat{q}/\partial\beta^2 = 2\,\partial\hat{\psi}/\partial Z, \tag{A 11}$$

and

$$\partial^2\hat{T}/\partial\beta^2 = \sigma S\,\partial\hat{\psi}/\partial\beta. \tag{A 12}$$

The boundary conditions at $\alpha = \beta = 0$ $(r = 1)$ are obtained by substituting (A 4)–(A 6) into the exact boundary conditions (14)–(17), with the following results:

$$q^{(I)} + \overline{q} = \omega, \tag{A 13}$$

$$\psi^{(I)} + \overline{\psi} + \hat{\psi} = 0, \tag{A 14}$$

$$\partial\hat{\psi}/\partial\beta = 0 \tag{A 15}$$

and

$$T^{(I)} = 0 \quad \text{(fixed temperature)} \tag{A 16a}$$

or

$$\partial T^{(I)}/\partial r = \partial\overline{T}/\partial\alpha + \partial\hat{T}/\partial\beta \quad \text{(insulated)}. \tag{A 16b}$$

Consider first the case of a fixed-temperature side wall. Then equation (A 16a) is a condition on the interior flow. As we have seen in § 3, the solution of the interior flow equations is unique when the conditions (A 1), (A 2) and (A 16a) are imposed. We indicate briefly how the boundary-layer corrections can be computed. By combining (A 7)–(A 9), we get

$$\frac{\partial\overline{q}}{\partial t} = \frac{\partial^2\overline{q}}{\partial\alpha^2} + \frac{4}{\sigma S}\frac{\partial^2\overline{q}}{\partial Z^2}. \tag{A 17}$$

The initial condition for $\overline{q}$ is

$$\overline{q}(\alpha, Z, 0) = 0 \tag{A 18}$$

and the boundary condition follows from (A 13)

$$\overline{q}(0, Z, t) = \omega - q^{(I)}(1, Z, t), \tag{A 19}$$

where $q^{(I)}$ is known since the interior solution is already determined. We also need end-point conditions. Since the $E^{\frac{1}{4}}$ layer is much thicker than the Ekman layer, conditions of the type (A 1) hold and, since

$$\overline{\psi} = (\sigma S)^{-1}\frac{\partial\overline{T}}{\partial\alpha} = -\frac{2}{\sigma S}\frac{\partial\overline{q}}{\partial Z}, \tag{A 20}$$

we get

$$\pm\frac{4}{\sigma S}\frac{\partial\overline{q}}{\partial Z} + \overline{q} = 0 \quad \text{at} \quad Z = \pm H. \tag{A 21}$$

Then $\overline{q}$ is determined uniquely by (A 17)–(A 19) and (A 21), $\overline{\psi}$ is determined uniquely by (A 20), and $\overline{T}$ is determined uniquely by (A 7).

Consider now the $E^{\frac{1}{3}}$ layer corrections. From (A 10)–(A 12), one can show that

$$\partial^4\hat{\psi}/\partial\beta^4 + \sigma S\hat{\psi} = 0. \tag{A 22}$$

The boundary conditions on $\hat{\psi}$ are (A 14) and (A 15) ($\psi^{(I)}$ and $\bar{\psi}$ are now known), and one can show that $\hat{\psi}$ is determined uniquely. From the integral of (A 10), we get

$$\hat{T} = -\partial^3\hat{\psi}/\partial\beta^3, \tag{A 23}$$

and $\hat{q}$ is determined uniquely by (A 11). Thus the boundary-layer corrections can be calculated, and no further conditions are imposed on the interior flow.

Consider now the case of an insulated side wall. We start with the thermal boundary condition (A 16*b*). From (A 20), we can express $\partial\bar{T}/\partial\alpha$ in terms of $\bar{\psi}$, and from the integral of (A 12), we can express $\partial\bar{T}/\partial\beta$ in terms of ψ. The condition (A 16*b*) then becomes

$$\partial T^{(I)}/\partial r = \sigma S\bar{\psi} + \sigma S\hat{\psi}. \tag{A 24}$$

Now we use (A 14) to get

$$\partial T^{(I)}/\partial r = -\sigma S\psi^{(I)}, \tag{A 25}$$

a condition on the interior flow. As shown in § 3, the interior flow is determined uniquely by (A 25) and the conditions (A 1), (A 2). The subsequent calculation of the boundary-layer corrections is identical to the fixed-temperature case already described.

Appendix B. General axisymmetric container

We consider here a general axisymmetric container and an arbitrary time variation of the container angular velocity. The equations for the interior flow are still (27)–(29). The boundary conditions are obtained by a boundary-layer analysis of the full equations (8)–(10) (for $a = \frac{1}{2}$). Since the analysis is similar to those in appendix A, we just quote the results. On a surface which is not vertical, the only possible boundary layer has thickness $E^{\frac{1}{2}}$. In this layer, the corrections to q_θ and ψ are $O(1)$ (in fact, q_θ and ψ satisfy the usual Ekman layer equations), and the correction to T is $O(E^{\frac{1}{2}})$. The conditions imposed on the interior flow are

$$2(\cos\theta)\,\psi = |\cos\theta|^{\frac{1}{2}}\,(q_\theta - \omega r), \tag{B 1}$$

and

$$T = 0 \quad \text{(fixed temperature)} \tag{B 2a}$$

or

$$\mathbf{n}\,.\,\nabla T + \sigma S(\sin\theta)\,\psi = 0 \quad \text{(insulated)}, \tag{B 2b}$$

where θ is the angle from the z axis to the exterior wall normal $\mathbf{n}$. These equations are also correct in the limiting cases of a horizontal wall ($\theta = 0$ or π) or a vertical wall ($\theta = \frac{1}{2}\pi$). We may contrast these results with those of Hsueh (1969) for the case $P = O(1)$. He found that the buoyancy force alters the structure of the Ekman layer unless $|\theta| < E^{\frac{1}{2}}$. In our case, the buoyancy force is unimportant in the Ekman layer.

The energy-dissipation integral (33) may be generalized for either of the boundary conditions (i) or (ii), or for the more general case where (i) is satisfied on part of the surface and (ii) on the remainder:

$$\frac{d}{dt}\iiint_V \{\tfrac{1}{2}\sigma T^2 + \tfrac{1}{2}S\sigma(q_\theta - \omega r)^2\}\,dV$$
$$= -\iiint_V (\operatorname{grad} T)^2\,dV - \iint_S S\sigma(q_\theta - \omega r)^2|\cos\theta|^{\frac{1}{2}}\,ds$$
$$-\sigma S\frac{\partial\omega}{\partial t}\iiint_V (q_\theta - \omega r)\,r\,dV, \tag{B 3}$$

where V is the volume bounded by the container surface S and ω is the container angular velocity, which may now depend on t. The uniqueness of the solution is established by applying (B 3) to the difference of any two solutions. This difference satisfies zero initial conditions, and the equations and boundary conditions with $\omega = 0$, so that uniqueness follows directly from (B 3). We also see that for any $\omega(t)$ such that $\omega \to \omega_0$ (a constant), as $t \to \infty$, the solution necessarily tends to a rigid-body rotation $q_\theta = \omega_0 r$ as $t \to \infty$. Since the quantities in (B 3) are of order one, the time lag between the interior and boundary angular velocities is of order one.

Appendix C. Expansions in Bessel functions

In §4, we used four different expansions in Bessel functions. We summarize here the relevant properties of these expansions. In all four cases, the set of functions can be generated by a Sturm–Liouville problem involving Bessel's equation of order p,

$$\frac{d}{dr}\left(r\frac{d\phi_n}{dr}\right) - \frac{p^2}{r}\phi_n = -\lambda_n r\phi_n, \tag{C 1}$$

with p being zero or one. In each case, the relevant interval is $0 < r < 1$, and the condition of regularity at $r = 0$ is imposed. The four problems differ in the choice of p and the bundary condition at $r = 1$. In each case the functions ϕ_n are orthogonal with respect to r and complete on $0 \leqslant r \leqslant 1$. The expansion of an arbitrary $G(r)$ is then

$$G(r) = \sum_n \frac{g_n}{N_n}\phi_n(r), \tag{C 2}$$

where

$$g_n = \int_0^1 G(r)\,\phi_n(r)\,r\,dr \tag{C 3}$$

and

$$N_n = \int_0^1 [\phi_n(r)]^2\,r\,dr. \tag{C 4}$$

We now consider the four cases. The results stated below are easily proved with the aid of formulas from Watson (1958) and by integration by parts.

(i) For $p = 0$ and the boundary condition $\phi(1) = 0$, we get $\phi_n = J_0(\omega_n r)$ with $\lambda_n = \omega_n^2$ where ω_n is the nth positive root of J_0. The normalization is

$$N_n = \tfrac{1}{2}J_1^2(\omega_n). \tag{C 5}$$

If $G(r)$ is well-behaved and if $G(1) = 0$, then both G' and $(rG')'$ may be calculated by differentiating (C 2) termwise.

(ii) For $p = 1$ and the boundary condition $(r\phi)' = 0$ at $r = 1$, we get

$$\phi_n = J_1(\omega_n r) \quad \text{and} \quad \lambda_n = \omega_n^2,$$

where ω_n is again the nth positive root of J_0. The normalization is the same as (C 5). The series for any well-behaved G may be differentiated termwise to calculate $(rG)'$.

(iii) For $p = 0$ and the boundary condition $\phi' = 0$ at $r = 1$, we get $\phi_0 = 1$, $\lambda_0 = 0$ and $\phi_n = J_0(\gamma_n r)$ with $\lambda_n = \gamma_n^2$, where γ_n is the nth positive root of J_1. The normalization is

$$N_n = \tfrac{1}{2}J_0^2(\gamma_n). \tag{C 6}$$

The series for any well-behaved G may be differentiated termwise to calculate dG/dr.

(iv) For $p = 1$ and the boundary condition $\phi = 0$ at $r = 1$ we get $\phi_n = J_1(\gamma_n r)$ with $\lambda_n = \gamma_n^2$, where γ_n is the nth positive root of J_1. The normalization is the same as (C 6). If $G(1) = 0$, then $(rG)'$ may be calculated by termwise differentiation of the series for G.

By way of example, we prove the last result stated on termwise differentiation. We have

$$G = \sum_{n=1}^{\infty} \frac{g_n}{N_n} J_1(\gamma_n r) \tag{C 7}$$

with

$$g_n = \int_0^1 G(r)\, J_1(\gamma_n r)\, r\, dr \tag{C 8}$$

and N_n given by (C 6). We wish to show that

$$H = \frac{1}{r}\frac{d}{dr}(rG) = \sum_{n=1}^{\infty} \frac{\gamma_n g_n}{N_n} J_0(\gamma_n r). \tag{C 9}$$

We expand H in a series of type (iii):

$$H = \frac{h_0}{N_0} + \sum_{n=1}^{\infty} \frac{h_n}{N_n} J_0(\gamma_n r), \tag{C 10}$$

where

$$h_0 = \int_0^1 H(r)\, r\, dr = [rG]_0^1 = 0, \tag{C 11}$$

$$h_n = \int_0^1 \frac{d}{dr}(rG)\, J_0(\gamma_n r)\, dr, \tag{C 12}$$

and where N_n is still given by (C 6). An integration by parts in (C 12) leads to the desired result:

$$\begin{aligned} h_n &= [rGJ_0(\gamma_n r)]_0^1 - \int_0^1 G \frac{d}{dr} J_0(\gamma_n r)\, r\, dr \\ &= \gamma_n \int_0^1 G(r)\, J_1(\gamma_n r)\, r\, dr. \end{aligned} \tag{C 13}$$

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